Linear Transformation

PATNA WOMEN'S COLLEGE
Patna University
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Outline

1. Transformations
2. Linear Transformations

TRANSFORMATIONS I

DEFINITION

A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $x \in \mathbb{R}^n$, a vector $w = T(x)$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the domain of T, and $\mathbb{R}^m$ is called the codomain of T. The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
For $x \in \mathbb{R}^n$, the vector $w = T(x)$ in $\mathbb{R}^m$ is called image of x(under the action of T). The set of all images $w = T(x)$ is called the range of T.

EXAMPLE

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(x) = Ax$, $x \in \mathbb{R}^n$ and A is some $m \times n$ matrix. Then for each $x \in \mathbb{R}^n$, $T(x)$ is computed by Ax, where A is an $m \times n$ matrix i.e, $T(x)$ is in $\mathbb{R}^n$. Observe that the domain of T is $\mathbb{R}^n$, when A has n columns and the codomain of T is $\mathbb{R}^n$ when each column of A has m entries. The range of T is the set of all linear combinations of the columns of A, because each image $w = T(x)$ of x is of the form Ax. This mapping is called matrix transformation.

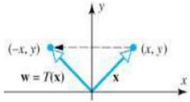
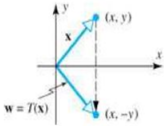
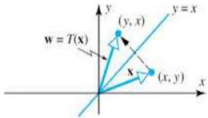
EXAMPLE

(Zero transformation)If 0 is the $m \times n$ zero matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$, then $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = 0x = \mathbf{0}$ maps every vector in $\mathbb{R}^n$ into the zero vector in $\mathbb{R}^m$. We call this T the zero transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

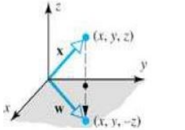
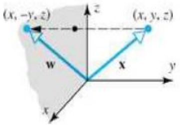
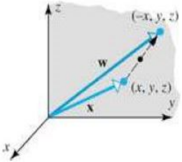
EXAMPLE

(Identity transformation)If I is the $n \times n$ identity matrix, then $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = Ix = x$ maps every vector in $\mathbb{R}^n$ into itself. We call this T the identity transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.

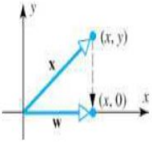
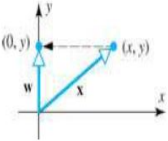
SOME MORE EXAMPLES I

Operator	Illustration	Equations
Reflection about the y-axis		$w_1 = -x$ $w_2 = y$
Reflection about the x-axis		$w_1 = x$ $w_2 = -y$
Reflection about the line $y = x$		$w_1 = y$ $w_2 = x$

Some more examples II

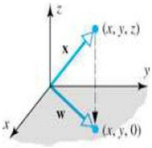
Reflection about the xy-plane		$w_1 = x$ $w_2 = y$ $w_3 = -z$
Reflection about the xz-plane		$w_1 = x$ $w_2 = -y$ $w_3 = z$
Reflection about the yz-plane		$w_1 = -x$ $w_2 = y$ $w_3 = z$

Some more examples III

Orthogonal projection on the x-axis		$w_1 = x$ $w_2 = 0$
Orthogonal projection on the y-axis		$w_1 = 0$ $w_2 = y$

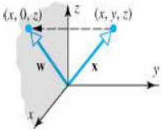
Some more examples IV

Orthogonal projection on the xy-plane



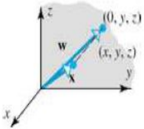
$$w_1 = x$$
$$w_2 = y$$
$$w_3 = 0$$

Orthogonal projection on the xz-plane



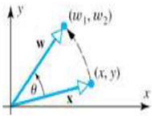
$$w_1 = x$$
$$w_2 = 0$$
$$w_3 = z$$

Orthogonal projection on the yz-plane



$$w_1 = 0$$
$$w_2 = y$$
$$w_3 = z$$

SOME MORE EXAMPLES V

Rotation through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$

Example

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$, so that

$$T(x) = Ax = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

1. Find $T(u)$, the image of u under the transformation T.
2. Find an $x \in \mathbb{R}^2$ whose image under T is b.
3. Is there more than one x whose image under T is b?

Solution

1. *Compute* $T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$.

2. *Solve* $T(x) = b$ *for* x. *That is, solve* $Ax = b$, *or*

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \tag{1}$$

Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Hence $x_1 = 1.5$, $x_2 = -0.5$, *and* $x = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$. *The image of this* x *under* T *is the given vector* b.

3. *Any* x *whose image under* T *is* b *must satisfy (1). From (2), it is clear that equation 1 has a unique solution. So there is exactly one* x *whose image is* b.

DEFINITION

A map(or function) $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation(or a linear map) if the following conditions are satisfied:

1. $T(x + y) = T(x) + T(y)$;
2. $T(\alpha.x) = \alpha.T(x)$,

where $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

REMARK

Conditions 1 and 2 given above are equivalent to the following
(a) $T(\alpha.x + \beta.y) = \alpha.T(x) + \beta.T(y)$

PROOF

If (a) holds, then for $\alpha = 1$ *and* $\beta = 1$*, we have (1) and for* $\beta = 0$*, we have (2).*
Conversely, if 1 and 2 hold, then

$$\begin{aligned} T(\alpha.x + \beta.y) &= T(\alpha.x) + T(\beta.y) \quad (by\ 1) \\ &= \alpha.T(x) + \beta.T(y) \quad (by\ 2) \end{aligned}$$

REMARK

If T *is a linear transformation, then* $T(0) = 0$*.*

PROOF

Since

$$
\begin{aligned}
& 0+0=0 \\
\Rightarrow \quad & T(0+0)=T(0) \\
\Rightarrow \quad & T(0)+T(0)=T(0) \\
\Rightarrow \quad & T(0)=0
\end{aligned}
$$

Some examples I

Example

Let $a \in \mathbb{R}$. Define $T_a : \mathbb{R} \to \mathbb{R}$ by $T_a = a.x$, where $x \in \mathbb{R}$. Then T is a linear map since for $x, y, \alpha \in \mathbb{R}$, we have

1.
$$\begin{aligned} T_a(x+y) &= a.(x+y) \\ &= a.x + a.y \\ &= T_a(x) + T_a(y); \end{aligned}$$

and

2.
$$\begin{aligned} T_a(\alpha.x) &= a.(\alpha.x) \\ &= (a.\alpha).x \\ &= (\alpha.a).x \\ &= \alpha.(a.x) \\ &= \alpha.T_a(x) \end{aligned}$$

Example

Consider the Euclidean n-space $\mathbb{R}^n$. Let $a \in \mathbb{R}$. Define $T_a : \mathbb{R}^n \to \mathbb{R}^n$ by $T_a = a.x$, where $x \in \mathbb{R}^n$. It can be verified that T is a linear transformation. Thus each $a \in \mathbb{R}$, give rise to a linear map T_a from $\mathbb{R}^n$ to $\mathbb{R}^n$. Note that if $a = 0$, then T_a is the zero map. It maps each element of $\mathbb{R}^n$ to 0.

Some examples II

Example

Consider the Euclidean spaces $\mathbb{R}^3$ and $\mathbb{R}^2$. Define

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

by $T(x, y, z) = (x, y)$, where $(x, y, z) \in \mathbb{R}^3$. Then T is a linear map follows from below:

1. For (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in \mathbb{R}^3$,

$$\begin{aligned} T((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x_1 + x_2, y_1 + y_2) \\ &= (x_1, y_1) + (x_2, y_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \end{aligned}$$

2. For $\alpha \in \mathbb{R}$ and $(x, y, z) \in \mathbb{R}_3$,

$$T(\alpha.(x, y, z)) = T(\alpha.x, \alpha.y, \alpha.z) = (\alpha.x, \alpha.y) = \alpha.(x, y) = \alpha.T(x, y, z)$$

Example

Define

$$T : \mathbb{R}^2 \to \mathbb{R}^3$$

by $T(x, y) = (x + 3, 2y, x + y)$, where $(x, y) \in \mathbb{R}^2$. Then T is not linear since $T(0, 0) = (3, 0, 0) \neq (0, 0, 0)$.

SOME EXAMPLES III

NOTE

It can be easily verified that all the transformations given in the diagrams above are linear transformations.

EXERCISES

Test whether the following maps are linear transformation or not:

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x)$, $(x, y) \in \mathbb{R}^2$.
2. $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + y + z, 2x - 3y + 4z)$, $(x, y, z) \in \mathbb{R}^3$.
3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (xy, x)$, $(x, y) \in \mathbb{R}^2$.
4. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (-y, x), (x, y) \in \mathbb{R}^2$.
5. $T : \mathbb{R} \to \mathbb{R}^2$ defined by $T(x) = (x, 0)$, $x \in \mathbb{R}$.
6. $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (2x - 3y, x + 4, 5y), (x, y) \in \mathbb{R}^2$.
7. $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(x) = Ax$, $x \in \mathbb{R}^n$ and A is some $m \times n$ matrix.