Matrix of a Linear Transformation

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OUTLINE

THE MATRIX OF A LINEAR TRANSFORMATION I

Whenever a linear transformation T arises geometrically or described in words, we usually want a "formula" for $T(x)$. In this section, we discuss the fact that every linear transfrmation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is actually a matrix transformation $x \mapsto Ax$ and that important properties of T are intimately related to properties of A. The key to finding A is to observe that T is completely determined by what it does to the columns of the $n \times n$ matrix I_n.

THE MATRIX OF A LINEAR TRANSFORMATION II

EXAMPLE

The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Suppose that T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $T(e_1) = (5, 1, 7)$ and $T(e_2) = (3, 8, 0)$. Then we write $x \in \mathbb{R}^2$ in the following form:

$$x = (x_1, x_2) = x_1(1, 0) + x_2(0, 1) = x_1 e_1 + x_2 e_2 \tag{1}$$

Since T is a linear transformation,

$$\begin{aligned} T(x) &= x_1 T(e_1) + x_2 T(e_2) \\ &= x_1(5, 1, 7) + x_2(3, 8, 0) \\ &= (5x_1 + 3x_2, x_1 + 8x_2, 7x_1) \end{aligned} \tag{2}$$

The steps from (1) to (2) explains why knowledge of $T(e_1)$ and $T(e_2)$ is sufficient to determine $T(x)$ fo any x. We can also write

$$T(x) = [T(e_1)\ \ T(e_2)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$$

THEOREM

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. then there exists a unique matrix A such that

$$T(x) = Ax, \text{ for all } x \in \mathbb{R}^n.$$

In fact, A is $m \times n$ whose j^{th} column is $T(e_j)$(in column form), where e_j is the j^{th} column of identity matrix in $\mathbb{R}^n$, i.e.,

$$A = [T(e_1), \ \ T(e_2) \ \ldots \ T(e_n)]. \tag{3}$$

PROOF

Write $x \in \mathbb{R}^n$ in the following form:

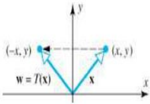
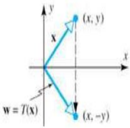
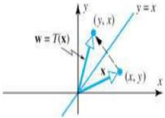
$$x = I_n x = \quad = [e_1 \;\; e_2 \ldots \; e_n]x$$
$$= x_1 e_1 + \ldots + x_n e_n$$

and by the linearity of T to compute

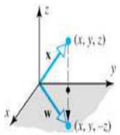
$$\begin{aligned} T(x) &= T(x_1 e_1 + \ldots + x_n e_n) \\ &= x_1 T(e_1) + \ldots + x_n T(e_n) \\ &= [T(e_1) \;\; T(e_2) \; \ldots \; T(e_n)] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = Ax \end{aligned}$$

NOTE

The matrix A in (3) is called the standard matrix for the linear transformation for the linear transformation T.

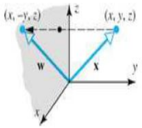
Operator	Illustration	Equations	Standard Matrix
Reflection about the y-axis		$w_1 = -x$ $w_2 = y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis		$w_1 = x$ $w_2 = -y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y = x$		$w_1 = y$ $w_2 = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Reflection about the xy-plane



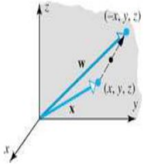
$$\begin{aligned} w_1 &= x \\ w_2 &= y \\ w_3 &= -z \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Reflection about the xz-plane



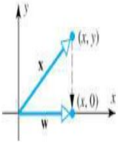
$$\begin{aligned} w_1 &= x \\ w_2 &= -y \\ w_3 &= z \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection about the yz-plane



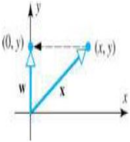
$$\begin{aligned} w_1 &= -x \\ w_2 &= y \\ w_3 &= z \end{aligned} \qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Orthogonal projection on the x-axis

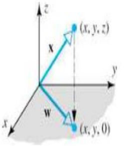
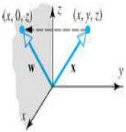
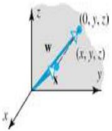


$$w_1 = x \qquad w_2 = 0 \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

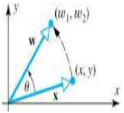
Orthogonal projection on the y-axis



$$w_1 = 0 \qquad w_2 = y \qquad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Orthogonal projection on the xy-plane		$w_1 = x$ $w_2 = y$ $w_3 = 0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection on the xz-plane		$w_1 = x$ $w_2 = 0$ $w_3 = z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection on the yz-plane		$w_1 = 0$ $w_2 = y$ $w_3 = z$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotation through an angle θ



$$w_1 = x\cos\theta - y\sin\theta \qquad w_2 = x\sin\theta + y\cos\theta \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

DEFINITION

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $b \in \mathbb{R}^m$ is image of at least one $x \in \mathbb{R}^n$.

NOTE

Note that T is onto $\mathbb{R}^m$ when the range of T is all of the codomain $\mathbb{R}^m$. That is, T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if, for each b in the codomain $\mathbb{R}^m$, there exist at least one solution of $T(x) = b$. The mapping T is not onto when there is some $b \in \mathbb{R}^m$ for which the equation $T(x) = b$ has no solution.

DEFINITION

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each b in $\mathbb{R}^m$ is the image of at most one x in $\mathbb{R}^n$.

NOTE

Note that T is one-to-one if, for each b in $\mathbb{R}^m$, the equation $T(x) = b$ has a unique solution or none at all. The mapping T is not one-to-one when some b in $\mathbb{R}^m$ is the image of more than one vector in $\mathbb{R}^n$.

Recall

1. *Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statement of they are all false.*
   1. *For each $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution.*
   2. *Each $b \in \mathbb{R}^m$ is a linear combination of the columns of A.*
   3. *The columns of A span $\mathbb{R}^m$.*
   4. *A has a pivot position in every row.*
2. *The columns of a matrix A are linearly independent if and only if the equation $Ax = 0$ has only the trivial solution.*

Example

Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Solution

Since A happens to be in Echelon form, we can see at once that A has a pivot position in each row. Then by result (1) in Recall, for each b in $\mathbb{R}^3$, the equation $Ax = b$ has a solution. In other words, the linear transformation T maps $\mathbb{R}^4$ ontoo $\mathbb{R}^3$. However, since the equation $Ax = b$ has a free variable(because there are four variables and only three basic variables), each b is the image of more than one x. That is, T is not one-to-one.

THEOREM

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution.

PROOF

Since T is linear, $T(0) = 0$. If T is one-to-one, then the equation $T(x) = 0$ has atmost one solution and hence only the trivial solution. If T is not one-to-one, then there is a b that is image of atleast two different vectors u and v in $\mathbb{R}^n$. Then $T(u) = b$ and $T(v) = b$. But then, since T is linear,

$$T(u - v) = T(u) - T(v) = b - b = 0$$

. The vector $u - v$ is not zero, since $u \neq v$. Hence the equation $T(x) = 0$ has more than one solution. So, either the two condition in the theorem are both true of they both are false.

THEOREM

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

1. *T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;*
2. *T is one-to-one if and only if the columns of A are linearly independent.*

Proof

1. *By result (1) of Recall, the columns of A span $\mathbb{R}^m$ if and only if for each b the equation $Ax = b$ has atleast one solution. This implies that the equation $T(x) = b$ has atleast one solution. This is true only if T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.*
2. *The equation $T(x) = 0$ and $Ax = 0$ are same except the notation. So by Theorem 6, T is one-to-one if and only if $Ax = 0$ has only the trivial solution. This happens if and only if the columns of A are linearly independent, by (2) of Recall.*

Note

Statement (1) of Theorem 7 is equivalent to the statement " T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if every vector in $\mathbb{R}^m$ is a linear combination of the columns of A".

Example

Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

Solution

The standard matrix A for the linear transformation T is(can be obtained by the method described before) the following:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$$

The columns of A are linearly independent because they are not multiples. By theorem 7(2), T is one-to-one. To decide if T is onto $\mathbb{R}^3$, examine the span of columns of A. Since A is 3×2, the columns of A span $\mathbb{R}^3$ if and only if A has 3 pivot positions, by Result (1) in Recall. This is impossible, since A has only 2 columns. So the columns of A do not span $\mathbb{R}^3$, and thus the associated linear transformation is not onto $\mathbb{R}^3$.