Date: / /
Page No.:

Lecture 4

Topics to be covered: Basis and dimension, Some important theorems.

Basis and dimension:

A set of vectors of a vector space V is said to be a basis of V if

(I) S is linearly independent

(II) V is generated or spanned by S, that is, $V = [S]$.

Dimension of a vector space is the number of elements in its basis.

Example (1) In the vector space $\mathbb{F}^n$, the set $S = \{e_1, e_2, \ldots, e_n\}$, where $e_i = (0, 0, \ldots, 1, 0 \ldots, 0)$ in which ith coordinate is 1 and the rest are zero, is a basis of $\mathbb{F}^n$.

Since we have already shown that $\mathbb{F}^n = [S]$ and we can easily verify that S is linearly independent

Therefore, $\boxed{\dim \mathbb{F}^n = n}$

(2) The infinite set $\{1, x, x^2, \ldots\}$ of polynomials in x over a field $\mathbb{F}$ is a basis of the vector space $\mathbb{F}[x]$.

We have already shown that $F[x]$ is spanned (or generated) by the set $\{1, x, x^2, \ldots\}$. So we only need to show that the set $\{1, x, x^2, \ldots\}$ is linearly independent. For this, we show that any finite subset of the set $\{1, x, x^2, \ldots\}$ is linearly independent.

Let $S_1 = \{x^{m_1}, x^{m_2}, \ldots, x^{m_r}\}$ be any finite subset of the set $\{1, x, x^2, \ldots\}$, where m_i's $(i = 1, 2, \ldots, r)$ are either 0 or any positive integer.

Assume that

$$\alpha_1 x^{m_1} + \alpha_2 x^{m_2} + \cdots + \alpha_r x^{m_r} = 0$$

$$\Rightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$$

$\Rightarrow$ S_1 is linearly independent

$\Rightarrow$ $\{1, x, x^2, \ldots\}$ is linearly independent.

Date: / /
Page No.:

$\dim \mathbb{F}[x] = \infty$.

(3) In the vector space $\mathbb{R}^3$ over the field $\mathbb{R}$, the set
$S = \{(1,0,0), (1,1,0), (1,1,1), (0,1,0)\}$
is not a basis of $\mathbb{R}^3$, since S is linearly dependent, through
$\mathbb{R}^3 = [S]$.

(4) If V is a finitely generated vector space over $\mathbb{F}$, then the set $S = \{x_1, x_2, \ldots x_n\}$ is a basis of V iff $x \in V$ can be expressed as a unique linear combination of elements of S.

Proof:- First, we assume that $S = \{x_1, x_2, \ldots, x_n\}$ is a basis of V. We show that $x \in V$ can be expressed as a unique linear combination of elements of S.

Since S is a basis of V, so by the definition of S, $x \in V$ can be expressed as a linear combination of elements of S. ($\because V = [S]$).

To prove the uniqueness, assume that

$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n$ for $x \in V$.

$\Rightarrow (\alpha_1 - \beta_1)x_1 + (\alpha_2 - \beta_2)x_2 + \cdots + (\alpha_n - \beta_n)x_n = 0$

$\Rightarrow \alpha_1 - \beta_1 = 0$, $\alpha_2 - \beta_2 = 0$, ⋮ $\alpha_n - \beta_n = 0$ } since the set $S = \{x_1, x_2, \ldots, x_n\}$ is linearly independent

$\Rightarrow \alpha_i = \beta_i, \; i = 1, 2, \ldots, n$

$\Rightarrow$ Representation of x is unique.

Conversely assume that $x \in V$ can be expressed as a unique linear combination of elements of S. In this case we have to show that S is a basis of V

By the assumption, we can clearly say that $V = [S]$. So we only need to show that S is linearly independent

Let $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0 = 0x_1 + 0x_2 + \cdots + 0x_n$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$

($\because$ the representation of '0' is unique)

Date: / /
Page No.:

$\Rightarrow S = \{x_1, x_2, \ldots, x_n\}$ is linearly independent.

Theorem: Every finitely generated vector space posses a basis.

Proof:- Let V be generated by the finite set

$$S = \{x_1, x_2, \ldots, x_n\}.$$

Let $S_1 = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ be a maximal subset of r linearly independent elements of S. We prove that S_1 is a basis of V, for which we only need to prove that $V = [S_1]$, since S_1 is linearly independent.

Now the set $\{x_{i_1}, x_{i_2}, \ldots x_{i_r}, x_{i_m}\}$, where $m = r+1, r+2, \ldots, n$ is linearly dependent, since S_1 is the largest subset of linearly independent elements of S.

Hence $\exists$ scalars $\alpha_1, \alpha_2, \ldots \alpha_r, k$ not all zero such that

$$\alpha_1 x_{i_1} + \alpha_2 x_{i_2} + \cdots + \alpha_r x_{i_r} + k x_{i_m} = 0$$

Scanned with CamScanner

Now $k \neq 0$, otherwise $\{x_{i_1}, x_{i_2}, \ldots x_{i_r}\}$ will be linearly dependent contradicting that S_1 is linearly independent. Hence from (1), we get $x_{i_m} = -k^{-1}\alpha_1 x_{i_1} - k^{-1}\alpha_2 x_{i_2} - \ldots - k^{-1}\alpha_r x_{i_r}$

Thus every x_{i_m}, for $m = r+1, r+2, \ldots n$ can be expressed as a linear combination of elements of S_1.

Hence every $x \in V$ can be expressed as a linear combination of elements of S_1 and so $V = [S_1]$ & S_1 is L.I. Therefore S_1 is a basis of V.

Theorem:- Any two bases of a finitely generated vector space have the same number of elements.

Proof: Let V have the following two bases

$B_1 = \{x_1, x_2, \ldots, x_m\}$ having m elements

$B_2 = \{y_1, y_2, \ldots, y_n\}$ having n elements

We have to show that $\underline{m = n}$.

Since V is generated by B_1, so $y_1 \in B_2 \subset V$ can be expressed as a linear combination of elements of B_1.

Hence the set $\{y_1, x_1, x_2, \ldots, x_m\}$ is linearly dependent.

Date: / /
Page No.:

⇒ ∃ an element ($\neq y_1$) which is a linear combination of preceding vectors.

Let it be x_i, deleting which we get the set

$S_1 = \{y_1, x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m\}$

Now it is clear that $x \in V$ will be a linear combination of S_1.

Hence $V = [S_1]$ i.e., V is generated by S_1.

So $y \in B_2$ can be expressed as a linear combination of elements of S_1 and hence the set

$\{y_2, y_1, x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m\}$

is linearly dependent.

⇒ ∃ an element of this set different from y_1 or y_2, say x_j, which is a linear combination of preceding vectors. Deleting x_j as before we get the set

$S_2 = \{y_2, y_1, x_1, \cdots x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_m\}$

which also generates V

Date: / /
Page No.:

Thus the successive generating sets S_1, S_2, etc. are obtained by excluding an x from B_1 and including a y from B_2 at each step.

Now if $m < n$. i.e., the number of elements of B_1 is less than the number of elements in B_2, then B_1 is exhausted before the set B_2, and we will get

$$S_m = \{y_m, y_{m-1}, \dots, y_2, y_1\}$$

as a generating set of V.

Now S_m is a proper subset of B_2, which is linearly dependent. So the set B_2, being a superset of a linearly dependent set, becomes linearly dependent, which is a contradiction.

Hence $m \nless n$, ie $m \geq n$ — (1)

Interchanging the roles of two bases, we can also prove that

$$n \geq m \quad \text{— (2)}$$

From (1) & (2), we get $m = n$.

Data: / /
Page No.:

Corollary:- If a vector space V is generated by a set of m vectors, then every linearly independent subset of V can contain utmost m vectors

Proof. Let $B_1 = \{x_1, x_2, \dots, x_m\}$ be a generating set of V and $B_2 = \{y_1, y_2, \dots, y_n\}$ be a linearly independent subset of V containing n vectors.

Then proceeding as in the previous theorem, we can prove that

$$n \leq m.$$

which proves that B_2 can have utmost m vectors.

Definition The invariant number of elements of different bases of a finitely generated vector space V is called the dimension of V and will be denoted by dim V.

Date: / /
Page No.:

Lecture 5

Topics to be covered: Some important theorem related to bases of a vector space

Theorem (Extension theorem).
Let V be a finite dimensional vector space over a field $\mathbb{F}$ and $S = \{x_1, x_2, \ldots, x_m\}$ be a set of linearly independent vectors of V. Then either S is a basis or can be extended by adjoining some more vectors to it.

Proof. Let $S = \{x_1, x_2, \ldots, x_m\}$ be a linearly independent set.

If $V = [S]$, then S is a basis.

Now suppose that S is a basis. Let $\dim V = n$ and $\{y_1, y_2, \ldots, y_n\}$ be a basis of V.

Now the set

$$S_1 = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots y_n\}$$

is linearly dependent since each x_i is a linear combination of the y_i's.

Date: / /
Page No.:

So $\exists$ some element of S which is a linear combination of the preceding ones. This can not be any of the x_i's since $\{x_i\}$ is linearly independent, so it must be some y_i say, deleting which we get

$$S_2 = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n\}$$

Now V is generated by S_2, since y_i is a linear combination of preceding one viz $x_1, x_2, \ldots x_m, y_1, \ldots, y_{i-1}$.

If S_2 is linearly independent, then it is the required extended basis.

Now S_2 will not ~~not~~ be linearly independent if S_2 contains more than elements than n. In this we ~~op~~ proceeding in the similar way as above till we get a linearly independent set generating V and containing $x_1, x_2, \ldots, x_m$ and some of the y's say $y_{i_1}, y_{i_2}, \ldots y_{i_{n-m}}$.

This last set $\{x_1, x_2, \ldots x_m, y_{i_1}, \ldots, y_{i_{n-m}}\}$ is the required extended basis of V.

Date: / /
Page No.:

Theorem:- Let V be a vector space over F, B be a non empty finite subset of non zero vectors of V. Then the following statements are equivalent:

(i) B is a basis of V.
(ii) B is a maximal set of linearly independent vectors of V.

(iii) B is the minimal set of generators of V.

Proof:- First we prove (i) ⇒ (ii)

Let B be a basis of V. So V is generated by B. Hence any L.I. subset of V cannot contain more elements than B, which is therefore a maximal set of linearly independent vector of V. Thus (i) ⇒ (ii).

Let $B = \{x_1, x_2, \ldots, x_n\}$ be a maximal set of L.I. vectors of V, ~~that~~ then if $x \in V$,
$$\{x, x_1, x_2, \ldots, x_n\}$$
is linearly dependent.

Date: / /
Page No.:

Now, we can prove that x is a linear combination of elements of B.
So B generates V, hence B is a basis of V. Any other generating set of V will contain more elements than that of B; so B is a minimal set of generators of V.

Thus (ii) ⇒ (i) and also (i) ⇒ (iii)

Now let $B = \{x_1, x_2, \ldots, x_n\}$ be a minimal set of generators of V.
Then B must be linearly independent otherwise B linearly Independent will imply that $\exists\, x_k \in B$ which is a linear combination of the preceding vectors $x_1, x_2, \ldots x_{k-1}$. Deleting x_k we get
$\{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots x_n\}$
to be a generating set of V, which contradicts that B is a minimal set of generators of V. Thus B is L.I and hence a basis of V.
Therefore (iii) ⇒ (i)

Date: / /
Page No.:

Theorem: A subspace W of an n-dimensional vector space V is of dimension $\leq n$.

Proof: Since $\dim V = n$, so any $(n+1)$ vectors of V are linearly dependent, in particular any $(n+1)$ elements of W are linearly dependent.

Thus, we can find a largest set of L.I. elements of W, viz

$S = \{w_1, w_2, \ldots, w_m\}$, where $m \leq n$

We prove $W = [S]$

If $w \in W$, then the set $\{w, w_1, \ldots, w_m\}$ is linearly dependent.

Hence $\alpha w + \alpha_1 w_1 + \cdots + \alpha_m w_m = 0$, where not all α's are 0

Now $\alpha \neq 0$, otherwise $\{w_1, w_2, \ldots w_n\}$ will be L.D., contrary to the supposition. And so

$w = -\alpha^{-1}(\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m)$

Hence $W = [S]$ & S is l.i.

Therefore, S is a basis of W and $\dim W = m \leq n$.