Lecture 8

Topics to be covered: Linear transformations, Some examples of linear transformations and their properties, kernel and image of a linear transformation, rank and nullity.

Linear transformations

Definition: Let V and U be vector spaces over the same field $\mathbb{F}$. A map $T: V \to U$ is called a linear transformation if the following conditions are satisfied.

(i) $T(x+y) = T(x) + T(y)$, $\forall\, x, y \in V$.
(that is, T preserves vector addition).

(ii) $T(\alpha x) = \alpha T(x)$, where $\alpha \in \mathbb{F}$, $x \in V$
(that is, T preserves scalar multiplication)

Note 1 From condition (i) above we can say that

$$T: (V, +) \to (U, +)$$

is a group homomorphism.

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Since in a group homomorphism identities correspond and inverses correspond, so

$T(0) = 0$ and $T(-x) = -T(x), \forall x \in V$

i.e., if T is a linear transformation from V to U, then the following conditions are satisfied:

(i) $T(0) = 0$.
(ii) $T(x) = -T(x), \forall x \in V$.

② The above two conditions in the definition of a linear transformation, can be combined into the following single condition:

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \quad \forall \alpha, \beta \in \mathbb{F}, \forall x, y \in V.$$

This can be generalized as follows:

If $x_1, x_2, \ldots, x_n \in V$, $\alpha_1, \alpha_2, \ldots \alpha_n \in \mathbb{F}$, then

$$T(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) = \alpha_1 T(x_1) + \alpha_2 T(x_2) + \cdots + \alpha_n T(x_n)$$

Def:- A linear transformation $T: V \to U$ is called an isomorphism if the map T is bijective i.e., both injective and surjective.

The two vector spaces are then said to be isomorphic and we write $V \cong U$.

Example:

(1) Let V, U be any vector spaces over the same field $\mathbb{F}$.

The map $O: V \to U$ defined by $O(x) = 0$ (zero), $\forall x \in V$ is a linear transformation (Prove), called zero linear transformation.

(2) Let V be a vector space over $\mathbb{F}$. The identity map

$$I: V \to V$$

defined by $I(x) = x$, $\forall x \in V$ is a linear transformation (Prove), which is called identity linear transformation. It is clear that the identity linear map I is an isomorphism, since it is bijective.

(3) Let C be the vector space of complex number over the field $\mathbb{R}$ of real numbers.

Then the map $T: C \to C$ defined by $T(z) = \bar{z}$ (complex conjugate of z), $\forall z \in C$

is a linear map.

Proof. If $\alpha_1, \alpha_2 \in \mathbb{R}$, $z_1, z_2 \in C$, then

$$T(\alpha_1 z_1 + \alpha_2 z_2) = \overline{\alpha_1 z_1 + \alpha_2 z_2}$$
$$= \alpha_1 \bar{z}_1 + \alpha_2 \bar{z}_2$$
$$= \alpha_1 T(z_1) + \alpha_2 T(z_2).$$

(4) The map $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x_1, x_2) = (x_1^2, x_2)$ is not linear.

For if $\alpha, \beta \in \mathbb{R}$, $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, then

$$T[\alpha(x_1, x_2) + \beta(y_1, y_2)]$$
$$= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$
$$= \left((\alpha x_1 + \beta y_1)^2, \alpha x_2 + \beta y_2\right)$$

Now,

$$\alpha T(x_1, x_2) + \beta T(y_1, y_2)$$
$$= \alpha(x_1^2, x_2) + \beta(y_1^2, y_2)$$
$$= (\alpha x_1^2 + \beta y_1^2, \alpha x_2 + \beta y_2)$$
$$\neq T[\alpha(x_1, x_2) + \beta(y_1, y_2)]$$

(5) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by
$$T(x,y) = (x+1, y+3).$$
Then T is not linear as
$$T(0,0) = (1,3) \neq (0,0).$$

Kernel and Image of a linear map

Let $T: V \to U$ be a linear transformation

Then the image of T written as $\mathrm{Im}\,T$ is defined as
$$\mathrm{Im}\,T = T(V) = \{u \in U \mid \exists\, v \in V, \text{ where } T(v) = u\}$$

Thus $u \in \mathrm{Im}\,T \Rightarrow \exists\, v \in V$ such that $T(v) = u$.

Hence $\boxed{\mathrm{Im}\,T \subseteq U}$

The kernel of T written as $\ker T$ is defined as
$$\ker T = \{x \in V \mid T(x) = 0\}$$

Since $T(0) = 0$, so $0 \in \ker T$, hence $\ker T \neq \phi$. It is clear that
$\boxed{\ker T \subseteq V}$

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Theorem Let $T: V \to U$ be a linear transformation. Then $\operatorname{Im} T$ is a subspace of U and $\ker T$ is a subspace of V.

Proof:- Since $T(0) = 0$, so $0 \in T(V)$, hence $\operatorname{Im} T$ is non empty.

Let $u_1, u_2 \in \operatorname{Im} T$ and $\alpha_1, \alpha_2 \in \mathbb{F}$. Then to prove that $\operatorname{Im} T$ is a subspace of U, we have to prove that

$$\alpha_1 u_1 + \alpha_2 u_2 \in \operatorname{Im} T$$

Since $u_1, u_2 \in \operatorname{Im} T \Rightarrow \exists\, v_1, v_2 \in V$ such that

$$T(v_1) = u_1, \quad T(v_2) = u_2$$

$$\text{Hence } \alpha_1 u_1 + \alpha_2 u_2 = \alpha_1 T(v_1) + \alpha_2 T(v_2)$$
$$= T(\alpha_1 v_1 + \alpha_2 v_2)$$
$$[\because T \text{ is linear}]$$

Hence $\alpha_1 u_1 + \alpha_2 u_2 \in \operatorname{Im} T$

$\left(\because v = \alpha_1 v_1 + \alpha_2 v_2 \in V \text{ such that } T(v) = \alpha_1 u_1 + \alpha_2 u_2\right)$

$\Rightarrow$ $\operatorname{Im} T$ is a subspace of U.

Next, we prove that ker T is a subspace of V.

Let $v_1, v_2 \in \ker T$, $\alpha_1, \alpha_2 \in \mathbb{F}$. Then to prove that ker T is a subspace of V, we have to prove that

$$\alpha_1 v_1 + \alpha_2 v_2 \in \ker T.$$

Now $v_1, v_2 \in \ker T \Rightarrow T(v_1) = 0$ and $T(v_2) = 0$

So, $T(\alpha_1 v_1 + \alpha_2 v_2)$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2)$$
$$= \alpha_1 . 0 + \alpha_2 . 0$$
$$= 0$$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 \in \ker T$

$\Rightarrow$ ker T is a subspace of V.